10 Useful Techniques

for Solving Olympiad Problems

Unfinished

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Contents

Introduction

It is a common remark when solving Olympiad problems to say that you do not know how to approach it as you have not seen a similar problem before, or that you do not know enough theory to solve it. This booklet is designed to act as a partial remedy, highlighting some of the basic tools and common types of problem found in Olympiad style papers.

I have tried to split the booklet into logical sections; however some techniques do not fit a specific topic as they can be used comprehensively. And if I were to explain every topic in depth the book would have to be many times longer; you can always do further research into the techniques mentioned!

But of course the best way to improve is to do problems; a plethora of problems; but having read this, I hope that you will feel a lot more confident on how to approach Olympiad problems.

Michael Ng

1

Useful Identities

There are many useful identities to use. Here are some very helpful ones;

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}) \tag{1.1}$$

Notice that when n is 2 we simply have the difference of two squares. Substituting $-y$ for y gives another identity where y is odd:

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}) \tag{1.2}$$

Another one; notice how all the $+$ change to $\times$ and vice versa on each side:

$$xyz + (x + y)(y + z)(z + x) = (x + y + z)(xy + yz + zx) \tag{1.3}$$

The following ones are less obvious:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \tag{1.4}$$

And normally you wouldn't expect to see sums of squares being factorised, but the *Sophie Germain* Identity is a beautiful exception; here it is with a quick derivation:

$$\begin{aligned} x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2) \end{aligned} \tag{1.5}$$

It turns out that the set S, consisting of all integers expressible as the sum of two squares, is closed under multiplication due to this beautiful identity below. It is very well worth checking it for yourself as it is very important!

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 \tag{1.6}$$

Another important and less well known observation is being able to spot squares of expressions; for example:

$$(1 + y + y^2)^2 = y^4 + 2y^3 + 3y^2 + 2y + 1 \tag{1.7}$$

Exercise 1

1. Prove the useful lemma that for a polynomial $P(x)$ with integer coefficients, and any two integers a, b,

   $$a - b \mid P(a) - P(b)$$

   (Hint: use Equation 1.1).

2. *(IMO 1984)* Find two integers a and b such that none of a, b or $a + b$ are divisible by 7, but $(a + b)^7 - a^7 - b^7$ is divisible by 7^7.

3. Prove that $5^m + 5^n$ is expressible as the sum of two squares if and only if $n - m$ is even, given that n and m are both positive integers.

4. Prove that there are infinitely many positive integers a such that $x^4 + a$ is composite.

2

Functional Equations

In the IMO problems there tend to be quite a few difficult functional equations in the ALgebra section. However without some techniques these sort of problems, especially since they are so abstract, may seem very challenging. Here I demonstrate some tricks and techniques.

In functional equations it is also of the utmost importance that *you check that the functions that you have found satisfy the original conditions* (in order to ensure that no extraneous ones were found along the way).

Tip 1 Substitute 0 or other constants, then try to make parts of the equation constant.

Example 1

Problem

(BMO1 2009) Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x)f(y) = f(x+y) + xy$ for all real numbers x, y.

Solution

Here I present a way to set out solutions to functional equations.

Let $P(x, y) = f(x)f(y) = f(x+y) + xy$. Now use **Tip 1**.

$$P(0,0) : f(0)^2 = f(0)$$

$$f(0)(f(0)-1) = 0 \implies f(0) = 0 \text{ or } 1.$$

But now set only $x = 0$.

$$P(0, y): \; f(0)f(y) = f(y)$$

Therefore if $f(0) = 0$ then $f(x) = 0$ for all x. Now consider if $f(0) = 1$.

Now we aim to use the known value of $f(0)$, so we set $y = -x$.

$$P(x, -x): \; f(x)f(-x) = 1 - x^2$$

But having 0 on the RHS is desirable, so consider $x = 1$ and $x = -1$. In both cases we find that

$$P(1, -1): \; f(1)f(-1) = 0$$

This gives us two cases, where $f(1) = 0$ and where $f(-1) = 0$. If $f(1) = 0$, then the original equation gives us $f(x+1) = -x$ so $f(x) = 1 - x$ and similarly if $f(-1) = 0$, then $f(x) = 1 + x$.

We have considered all the cases so we conclude that all possibilities are $f(x) = 0, f(x) = 1 - x, f(x) = 1 + x$. After checking we find that $x = 1, y = 1$ invalidates $f(x) = 0$, so our final answer is $\boxed{f(x) = 1 - x}$ or $\boxed{f(x) = 1 + x}$. ■

Tip 2 Take an educated guess of what the function could be at the start. This can be very helpful for finding good substitutions.

Tip 3 With functions defined on the natural numbers, induction can be very useful. This can also be extended to the rational numbers.

Example 2

Problem *(BMO1 2014)* Determine all functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the following condition: whenever a, b and c are positive integers such that $1/a + 1/b = 1/c$, then $1/f(a) + 1/f(b) = 1/f(c)$.

Solution

Using $(2,2,1)$, we find that $f(2) = 2f(1)$.

With this basis I shall now list some more tips for your own independent investigation. Some may need to be applied to the exercise below.

Tip 4 If there are functions within functions it is often a good idea to remove the function inside. For example if we had $f(x - f(x))$ we could try setting $x = f(x)$.

Tip 5 Investigate for bijectivity, surjectivity and injectivity of functions. If you are not familiar with these terms then you must learn what they mean. This is also often useful with problems in which one must prove that a function of such a form does not exist.

Tip 6 Remember that polynomials with odd degree have an infinite range.

Tip 7 With problems restricted to polynomials, using the degree of the polynomial is often the crucial step towards solving the problem!

Here are two important results for polynomials with integer coefficients.

Tip 8 Rational Root Theorem Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients, and let p/q be a rational root of $f(x)$ in its lowest terms (where p and q are coprime integers and $q \neq 0$). Then $q \mid a_n$ and $p \mid a_0$.

Tip 9 Gauss' Lemma If a polynomial with integer coefficients is factorizable into (non-scalar) polynomials with rational coefficients, then it is factorizable into (non-scalar) polynomials with integer coefficients.

Exercise 2

1. Prove that there do not exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + f(y)) = y^2 + g(x) \text{ for all } x, y \in \mathbb{R}$$

2. FUNCTIONS ARE FUN

3

Techniques for Inequalities

3.1 Sums of Squares and the AM-GM Inequality

Most inequalities at Olympiad level can be solved using some cunning applications of the AM-GM inequality or sometimes simply sums of squares. But be aware that *the AM-GM inequality only works for non-negative (real) numbers.* That's because square rooting negative products leads into complex numbers, whereas most problems will be limited to real numbers.

Tip 1 Completing the square is very useful for some types of problem.

Tip 2 Consider parts of the inequality then multiply or add them together to give the desired one.

Example 1

Problem

Prove that:

$$x^2 + y^2 + z^2 \geq xy + yz + zx$$

Solution

Notice that $(x - y)^2 \geq 0 \implies x^2 + y^2 \geq 2xy$, but then we use the spectacular step of adding this to the other two permutations with y, z and z, x. Dividing by two then yields the required result. ■

Tip 3 With fractions, the AM-GM inequality can be used as follows:

$$\frac{x^2}{y} + y \geq 2x$$

Tip 4 The AM-GM inequality can be *weighted* to gain higher powers. Do this by **repeating terms**.

Tip 4 is of the utmost importance and can be very useful.

Example 2

Problem

For positive reals a, b, c where $a + b + c = 1$, prove that

$$ab^2c^3 \leq 108$$

Solution

Use the AM-GM inequality as follows:

$$\begin{aligned} 1 = a + b + c &= a + \frac{b}{2} + \frac{b}{2} + \frac{c}{3} + \frac{c}{3} + \frac{c}{3} \\ &\geq \sqrt[6]{\frac{ab^2c^3}{108}} \\ \therefore ab^2c^3 \leq 108 \ \blacksquare \end{aligned}$$

Notice how we split the b and c to get the required powers. In fact we can generalise to give the result below.

3.2 Weighted AM-GM Inequality

If $a_1, a_2, \ldots, a_n$ are non-negative real numbers, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-negative real numbers (the "weights") which sum to 1, then we have:

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \geq a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n},$$

or in compact notation:

$$\sum_{i=1}^{n} \lambda_i a_i \geq \prod_{i=1}^{n} a_i^{\lambda_i}.$$

Equality holds if and only if all a_i are equal to one another, for all integers i such that $\lambda_i \neq 0$.

3.3 Using the Cauchy-Schwarz Inequality

There is a very useful form of the *Cauchy Schwarz* Inequality that is helpful for spotting ways into solutions for inequalities. It is known as the *Engel Form*:

$$\frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n}$$

for positive real $a_1 \cdots a_n, b_1 \cdots b_n$.

Although one might argue that it is a direct consequence of the Cauchy-Schwarz Inequality, it makes spotting applications a lot easier.

3.4 Other Useful Techniques

3.4.1 Chebyshev's Sum Inequality

For similarly sorted sequences $\{a_1, \cdots, a_n\}$, $\{b_1, \cdots, b_n\}$, the following inequality holds:

$$\frac{\Sigma a_i b_i}{n} \geq \left(\frac{\Sigma a_i}{n}\right)\left(\frac{\Sigma b_i}{n}\right)$$

and the inequality sign is switched direction if the sequences are oppositely sorted.

This inequality is very useful for splitting sums with products and can be proved using the Rearrangement Inequality.

3.4.2 A Very Useful Lemma

Unfortunately this lemma does not have an well-known name, although some call it the Generalised Hölder's Inequality. However in Olympiad contests it is best to state it with its proof (which is not too difficult) in order to use it.

For positive reals $a_{i_j}, 1 \leq i \leq m, 1 \leq j \leq n$,

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i_j}^m\right) \geq \left(\sum_{j=1}^{n}\prod_{i=1}^{m} a_{i_j}\right)^m$$

This seems to be quite daunting, but it is not when you get used to it. For the proof it is much easier to prove a specific case; let us prove the case when $m = 3, n = 2$; that is,

$$(a^3+b^3)(c^3+d^3)(e^3+f^3) \geq (ace+bdf)^3$$

$$\begin{aligned} 3 &= \frac{a^3+b^3}{a^3+b^3} + \frac{c^3+d^3}{c^3+d^3} + \frac{e^3+f^3}{e^3+f^3} \\ &= \frac{a^3}{a^3+b^3} + \frac{c^3}{c^3+d^3} + \frac{e^3}{e^3+f^3} + \frac{b^3}{a^3+b^3} + \frac{d^3}{c^3+d^3} + \frac{f^3}{e^3+f^3} \\ &\geq \frac{ace}{\sqrt[3]{(a^3+b^3)(c^3+d^3)(e^3+f^3)}} + \frac{bdf}{\sqrt[3]{(a^3+b^3)(c^3+d^3)(e^3+f^3)}} \end{aligned}$$

And with some tidying up,

$$(a^3 + b^3)(c^3 + d^3)(e^3 + f^3) \geq (ace + bdf)^3 \blacksquare$$

From the AM-GM step it follows that equality holds if and only if $(a, c, e) = \lambda(b, d, f)$ for some positive real λ.

3.4.3 Others

Others can be found in my sheet on Inequalities on my website: `http://michaelng126.x10.mx/maths/reference/Inequalities.pdf`

Particularly useful inequalities include **Jensen's Inequality** and **Schur's Inequality**.

Substitution may also come into use, but that is another story altogether. Reciprocal substitutions are often helpful, but trigonometric ones are also extremely powerful. See Section 2.1.3 for more on this matter.

Exercise 1

1. I love inequalities!

4

Vieta's Formulas

Vieta's formulas are all to do with relationships between the roots of polynomials. In this section you will discover why they are so useful.

If the term with the highest degree in a polynomial has coefficient 1, a polynomial is said to be **monic**. For example, $x^2 - x + 1$, but not $3x^2 - 4x + 1$.

If a monic quadratic equation is of the form

$$(x - \alpha)(x - \beta) = 0$$

then its roots are α, β. That is obvious. But now expand to give

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \tag{4.1}$$

So for $x^2 - ax + b = 0$, the sum of the roots is simply a and product of the roots b. A spectacular result!

Example 1

Problem

Given the equation $x^2 - 5x + 9 = 0$ has two solutions α and β, find $\alpha^2 + \beta^2$.

Solution

Using our result, we see that $\alpha + \beta = 5$ and that $\alpha\beta = 9$. So

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 5^2 - 2 \times 9 = 25 - 18 = 7 \blacksquare$$

We could have directly found $\alpha, \beta = \frac{5 \pm i\sqrt{11}}{2}$, but finding $\alpha^2 + \beta^2$ could then lead to calculation errors; using such a result simplifies the problem greatly and reduces risk of error.

From Equation 4.1, a direct consequence is Vieta's Formula for Quadratics:

Given that $ax^2 + bx + c = 0$ has two roots r_1 and r_2, then:

$$r_1 + r_2 = -\frac{b}{a}, \quad r_1 r_2 = \frac{c}{a} \tag{4.2}$$

(Notice that we divide by a to make the quadratic monic ($x^2 + \frac{b}{a}x + \frac{c}{a} = 0$), then the result follows.)

4.1 Generalization to Higher Degrees

Using a similar process found for Equation 4.1, one can prove Vieta's Formulas for polynomials of higher degrees. And so:

Vieta's Formula Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients and degree n, having complex roots $r_n, r_{n-1}, \cdots, r_1$. Then for any integer $0 \le k \le n$:

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} r_{i_1} r_{i_2} \cdots r_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}.$$

This may seem daunting but really, if you wish to apply it, it is much easier to derive the formulas for your specific case using a method like that shown for Equation 4.1.

To demonstrate, it is not difficult to show that for the equation $x^3 - ax^2 + bx - c = 0$ with three (with some perhaps complex) roots r_1, r_2, r_3, $r_1 r_2 + r_2 r_3 + r_3 r_1 = b$.

4.2 Tips and Warnings

Vieta's Formulas are especially powerful for finding **symmetric** quantities relating the roots of equations. Use them with the factorization and identities found in **Section 1**.

However one must be careful when using Vieta's Formulas. Three common mistakes:

- Do not forget the signs, especially the minuses (e.g. for x^2+ax+b, the sum of the roots is $-a$).
- Do not forget to divide by the coefficient of the term with the highest degree.
- **Important** Vieta's Formulas apply to *all* roots, whether real or complex; this is shown in the **Example** above. So when a problem asks for real roots, you may be unable to apply Vieta's Formulas directly.

Exercise 4

1. *(Junior Balkan MO 1999)* Let a, b, c, x, y be five real numbers such that $a^3 + ax + y = 0$, $b^3 + bx + y = 0$ and $c^3 + cx + y = 0$. If a, b, c are all distinct numbers prove that their sum is zero.
2. I love these cool results!

5

Extraneous Roots

Solve:

$$\sqrt{x+4} = x - 2$$

Did you square both sides and get $x = 0, 5$? Is 0 really a solution?

It is common to solve an equation only to find that one solution does not solve the original equation (as it is outside the domain of the original function). This is known as an extraneous solution, and accentuates the importance of **checking solutions work**, much like functional equations.

This often happens when we multiply by a variable factor or square both sides of an equation as in the above example, along with many other possible causes.

That mistake is well known, however some problems can be much more devious. Try this problem for yourself first.

Problem

Find all pairs of real numbers a, b such that

$$\frac{ax+b}{1-x} = x + 3$$

has exactly one solution; $x = 5$.

Fallacious Solution

Seeing 'one solution', we realise that when we form a quadratic, having multiplied both sides by $1-x$, we need values of a, b so that this quadratic is precisely $x^2-10x+25=0$. And so we find that $a=8$,$b=28$. Substituting these values in works, so we get $a=8, b=28$. ■

This is incorrect because we have actually missed a pair of solutions! The mistake is in the very first step.

'One solution' could also mean the quadratic $(x-1)(x-5)=0$; that is, with 5 and 1 as an extraneous root! How crafty! From this equation we find another solution $a=-8, b=8$ (see this for yourself after some simple manipulation).

The main reason for this fallacy is assuming that 'one solution' for the original equation occurs in the same conditions as 'one solution' for the resulting quadratic.

So you must be very careful when presented with such problems.

Exercise 5

1. (*2002 Czech and Slovak Olympiad Q4*)

   Find all pairs of real numbers a, b for which the equation in the domain of the real numbers

   $$\frac{ax^2-24x+b}{x^2-1}=x$$

   has two solutions and the sum of them equals 12.

6

Trigonometry

6.1 Trigonometric Identities

It really is of the utmost importance to recall trigonometric identities easily. I list them (as proofs of them can be found from other sources). If you haven't proved them before, it is extremely beneficial to do so.

$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A + 1 = \sec^2 A$$
$$1 + \cot^2 A = \csc^2 A$$

	$-A$	$90 - A$	$180 - A$
sin	$-\sin A$	$\cos A$	$\sin A$
cos	$\cos A$	$\sin A$	$-\cos A$
tan	$-\tan A$	$\cot A$	$-\tan A$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

If we let $A = B$ then we get the double-angle formulae:

$$\sin(2A) = 2 \sin A \cos A$$

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A \\ \tan(2A) &= \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

Notice that we get lots of the difference of two squares appearing, which leads to nice problems.

The factor formulae are also very important, as they convert sums of trigonometric functions to products and vice versa.

$$\begin{aligned}\sin(A+B)+\sin(A-B) &= 2\sin A\cos B \\ \sin(A+B)-\sin(A-B) &= 2\cos A\sin B \\ \cos(A+B)+\cos(A-B) &= 2\cos A\cos B \\ \cos(A+B)-\cos(A-B) &= -2\sin A\sin B\end{aligned}$$

Exercise 6a

1. Prove that $\cos\theta + \sin\theta = \pm\sqrt{1+\sin 2\theta}$.
2. *(Original)* Solve
$$\frac{\operatorname{cosec} x - \cot x}{\sec y + \tan y} - \frac{\sec y - \tan y}{\operatorname{cosec} x + \cot x} = 1$$
for real $0^\circ \leq x, y \leq 360^\circ$.
3. *(Original)* Solve
$$\sqrt{3}\cos\theta + \cos 2\theta + 2\sin^2\theta = \sin\theta$$
for $0 \leq \theta \leq 2\pi$.

4. *(Original)*

   a) Show that

   $$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

   b) Hence or otherwise, solve

   $$2\sin\theta\,(\sin 2\theta + \sin 4\theta + \sin 6\theta + \cdots + \sin 14\theta) = \cos\theta - \frac{1}{2}$$

   for $0^\circ \le \theta \le 24^\circ$.

5. Given that

   $$\sin(70^\circ - \theta) = 2\cos 50^\circ \sin\theta$$

   a) Using $\sin 20^\circ$ and $\cos 20^\circ$, find an expression for $\tan\theta$ and hence the value of $\tan\theta$.

   b) Therefore solve the equation for $0^\circ \le \theta \le 360^\circ$.

6.2 Trigonometric Substitutions

Trigonometric substitutions are not only useful in some beautiful integrals, but also in Olympiad problems as well. The general tip is to use them when expressions look like the identities; $\tan A \pm B$ is one that often appears a lot. For example,

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Example Prove that in any seven distinct real numbers we can always choose two x, y say, such that:

$$0 < \frac{x - y}{1 + xy} < \sqrt{3}$$

Solution

Take $x = \tan\alpha$ and $y = \tan\beta$ for some real numbers α, β where $-\frac{\pi}{2} < \alpha < \beta < \frac{\pi}{2}$. It is clear that this is possible and unique as tan is a one to one function in that interval. Then by the Pigeon-hole Principle some two numbers (WLOG assume α and β) will have a (positive) difference less than $\frac{\pi}{6}$; that is to say:

$$0 < \beta - \alpha < \frac{\pi}{6}$$

As tan increases in our interval it is clear that we can take the tangent function of both sides, which yields the desired result as required. ■

6.3 Trigonometric Substitutions in Inequalities

Exercise 6b

1. Trigonometry is fun!

7

Finding Loci

Olympiad loci questions are often quite challenging. I remember that when doing the BMO2 in 2015, failing to spot the last step of a locus question cost me seven marks! Reflecting (and going through a lot more locus questions), one tip shines out from the rest:

Look for quantities or objects that are **fixed** (or **constant**).

Had I known this, perhaps I would have solved the question in the BMO2! In fact I have posed the problem in the exercise at the end of this section.

Example 1

Problem *(Czech and Slovak Olympiad 2002 Q2)*

Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square $ABCD$. Find the locus of the midpoints of the sides KL of all such triangles KLM.

Solution

Let E be the midpoint of KM. But by the converse of opposite angles in a cyclic quadrilateral summing to $180°$, $BLEK, CMEL$ are cyclic.

Now by angles in the same segment, $\angle LBE = \angle LKE = 60°$, and similarly $\angle LCE = 60°$. Therefore BCE is equilateral, and so E **is a fixed point**.

Let S be the midpoint of KL. But now see that S is the circumcentre of BLK and therefore the circumcentre of $BLEK$. Therefore S always

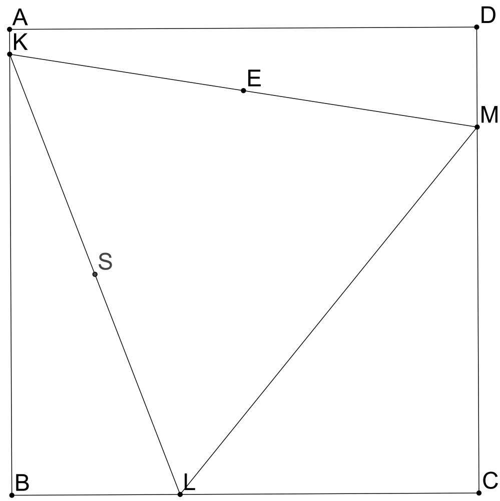

Figure 7.1: Czech and Slovak Olympiad 2002 Q2

lies on the perpendicular bisector of the **fixed segment** BE, and so the locus of S is a line segment on the perpendicular bisector of BE. ■

Example 2

Problem

A variable triangle has a fixed incircle. Given that the circumradius is constant, find the locus of the circumcentre.

Solution

This problem at first sight seems to be quite daunting. However there is an elegant solution which can be deduced very quickly using the idea of fixed points and quantities.

First identify what is fixed or constant. We know that I and r are fixed from the fact that is incircle is fixed. We know that R is constant. We need to find a relationship between these and O. But now this is obvious! It is well known that:

$$OI^2 = R^2 - 2Rr$$

Now the RHS is constant and therefore OI is constant. The solution immediately follows: the locus of O is a circle of radius $\sqrt{R^2 - 2Rr}$ centred at I. ■

Using this idea you should be able to solve most loci problems.

Exercise 7

1. Given two fixed points A, B, find the locus of the point P such that:

   a) $AP = BP$

   b) $AP = 2BP$

   c) What if the 2 was changed to a 3? Or any constant $\neq 1$?

   In fact these are known as **Apollonius —s** (fill in with your answer).

2. *(BMO 2010/11 Q5)* Circles S_1 and S_2 meet at L and M. Let P be a point on S_2. Let PL and PM meet S_1 again at Q and R respectively. The lines QM and RL meet at K. Show that, as P varies on S_2, K lies on a fixed circle.

3. *(BMO2 2015 Q3)* Two circles touch one another internally at A. A variable chord PQ of the outer circle touches the inner circle. Prove that the locus of the incentre of triangle AQP is another circle touching the given circles at A.

8

Catalan Numbers

8.1 An Explicit Formula

So far we have found a recurrence relation for the Catalan numbers, but it is also useful to have an explicit formula. In fact it is surprisingly simple:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

There are many proofs that show why the recurrence relation yields such an explicit formula, including ones with calculus, lattice paths and others. However my favourite is this one, which uses the fact that the number of triangulations of a polygon with $n+2$ sides is equal to C_n.

Proof

Let P be a polygon with $n+1$ sides, and Q be a polygon with $n+2$ sides. Let both polygons WLOG have a fixed base which lets us ignore rotations of the triangulations.

Define a '*special* triangulation of P' to be a triangulation of P where one edge has been marked, and furthermore this edge has had one of its ends marked (left polygon in the diagram).

Now all triangulations of P have $2(n+1) - 3 = 2n - 1$ sides. Then it is clear that there are $2(2n-1)C_{n-1}$ special triangulations of P (accounting for the fact that line segments have two ends).

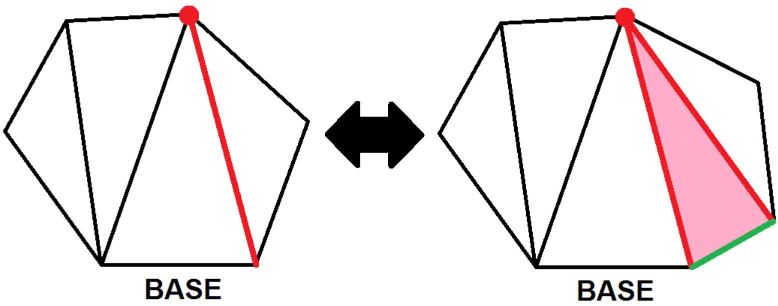

Figure 8.1: Bijecting Special Triangulations

Similarly define a '*special* triangulation of Q' to be a triangulation of Q where one of the sides of the polygon that is NOT the base has been marked (green side in the right polygon in the diagram).

Therefore there are $(n+1)C_n$ special triangulations of Q.

We will now establish a bijection between the special triangulations of P and the special triangulations of Q.

Given any special triangulation of P, take the marked edge and 'open' it up by rotating on the marked end as shown in the diagram to give a new side. Mark this new side and we now have a special triangulation of Q.

Given any special triangulation of Q, take the marked side and now collapse its triangle. It is clear that this yields a special triangulation of P.

Therefore there exists a bijection between the special triangulations of

P and the special triangulations of Q, as required. And so

$$C_n = \frac{2(2n-1)}{(n+1)} C_{n-1}$$

Furthermore as $C_0 = 1$ a simple induction shows that

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

and we are done!

9

Squares and Primes

In this chapter I have grouped several results concerning squares and primes. They are all very elegant results, with illuminating proofs that utilize methods that can be used in Olympiad problems.

9.1 The Factors of $4k^2 + 1$

Take any number of the form $4k^2 + 1$, where k is any integer. Now list out its factors and repeat for other k. What do you notice about all of the factors? In fact they are all of the form $4m + 1$ (where m is an integer).

Proof

It is easy to see that it suffices to show that all prime factors of $4k^2 + 1$ are of the form $4m + 1$. As $4k^2 + 1$ is odd, all of its prime factors must either be of the form $4m + 1$ or $4m + 3$. Let q be a prime of the form $4m + 3$ and suppose that it divides $4k^2 + 1$.

Therefore

$$4k^2 \equiv -1 \bmod q$$

But then as $4k^2 = (2k)^2$,

$$\begin{aligned}(4k^2)^{\frac{q-1}{2}} &\equiv ((2k)^2)^{\frac{q-1}{2}} \\ &\equiv ((2k)^2)^{2m+1} \\ &\equiv (-1)^{2m+1} \equiv -1 \bmod q\end{aligned}$$

However Fermat's Little Theorem shows that

$$((2k)^2)^{\frac{q-1}{2}} \equiv (2k)^{q-1} \not\equiv -1 \bmod q$$

which is a contradiction. ■