Inequalities

Michael Ng

1. Sums of Squares

- Completing the Square
- Adding or Multiplying related inequalities

Prove that:

1) $x^2 - 4x + 5 > 0$ for all real x
2) $x^2 - 4y + y^2 - 6z + z^2 - 2x + 14 \geq 0$
3) $(b+c-a)(c+a-b)(a+b-c) \leq abc$ where a, b, c are the sides of a triangle
4) $x^2 + y^2 + z^2 \geq xy + yz + zx$ for positive x, y, z

2. The AM-GM Inequality

- Applying this to multiple parts of inequalities (splitting up)
- Fractional Inequalities and cancelling out
- Repeating the same term

Prove that:

1) $(a+b)(b+c)(c+a) \geq 8abc$ for all positive a, b, c
2) $\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{d} + \frac{d^2}{a} \geq a + b + c + d$ for positive a, b, c, d
3) $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$ for positive a, b, c

3. Other Power Mean Inequality derivations

- AM-QM for square roots
- AM-HM for many reciprocals
- Beware of using the GM when variables can be negative!

Prove that:

1) $x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}$ where x, y, z are positive and $x^2 + y^2 + z^2 = 1$ (BMO 2002)

4. Cauchy-Schwarz Inequality

- Useful Forms (first – common version; second – Engel Form; third – good with square roots)

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2$$

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n}$$

$$\sqrt{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)} \geq \sqrt{a_1 b_1} + \sqrt{a_2 b_2} + \cdots + \sqrt{a_n b_n}$$

- The Engel Form is particularly useful with manipulating separate fractions into one fraction (see 4.2)
- Substituting ones

Prove that:

1) $3(a^2 + b^2 + c^2) \geq (a+b+c)^2$
2) $\frac{9}{a} + \frac{16}{b} + \frac{4}{c} + \frac{1}{d} \geq \frac{100}{a+b+c+d}$
3) $\sqrt{\frac{x+y}{x+y+z}} + \sqrt{\frac{y+z}{x+y+z}} + \sqrt{\frac{z+x}{x+y+z}} \leq \sqrt{6}$

5. Hölder's Inequality

- Useful for lots of the same power (generalisation of Cauchy-Schwarz)
- Form for nine variables:

$$(a^3 + b^3 + c^3)(d^3 + e^3 + f^3)(g^3 + h^3 + i^3) \geq (adg + beh + cfi)^3$$

Prove that:

1) $\sum \frac{a}{\sqrt{a^2 + 8bc}} \geq 1$ where a, b, c are positive real numbers. (IMO 2001)

6. Evaluating Minima and Maxima

- Differentiation may be possible for one variable, but for more than one the knowledge of inequalities is incredibly useful.
- We always aim for *one side of the inequality being a constant* in these types of problems.
- You *must* show that the minimum/maximum is possible (normally at equality); otherwise the problem is incomplete.
- Splitting variables and repeating them can be very useful.

Where a, b, c are positive real numbers and x, y, z are real,

1) Find the maximum area of a rectangle with perimeter 12.
2) Minimise:
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$
3) Maximise:
$$ab^2c^3 \text{ where } a^2+b^2+c^2=1$$
4) Minimise: (MOG 2014)
$$\frac{a^2}{b}+\frac{b}{c^2}+\frac{c}{a}$$
5) Minimise: (BMO2 2007)
$$x^2+y^2+z^2 \text{ where } x^3+y^3+z^3-3xyz=1$$

7. Other Inequalities and Techniques

- **Rearrangement Inequality**
  Used when two (or more) sorted sequences are multiplied together. It is very useful and 'strong'.
- **Schur's Inequality**
  $a^n(a-b)(a-c)+b^n(b-c)(b-a)+c^n(c-a)(c-b)\geq 0$ for positive a, b, c, n.
  Equality holds when $a=b=c$ or permutations of $a=b, c=0$.
  Used rarely (IMO).
- **Jensen's Inequality**
  For a convex function ($f''(x)\geq 0$):
$$\frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n}\geq f(\frac{a_1+a_2+\cdots+a_n}{n})$$
  Incredibly useful and powerful.
- **Ravi Transformation (use 1.3 as an example)**
- **Muirhead Inequality (unfortunately not recognised in many Olympiad competitions)**
- **Lagrange Multipliers for finding minima and maxima**
- **Homogenization (sometimes using condition)**
- **Substitution**
  Trigonometric substitutions may be helpful, as well as using reciprocals for example.

8. Additional Problems

1) For $x, y, z > 0$, prove that:
$$9(a^3+b^3+c^3)\geq(a+b+c)^3$$
2) For $x, y, z > 0$ where $x^2+y^2+z^2=1$, maximise:
$$\sqrt{10}xy+yz$$
3) For $x, y, z > 0$ where $x+y+z=3$, prove that:
$$xy+yz+zx\leq 3 \text{ and } x^2+y^2+z^2\geq 3$$
4) For $x, y, z > 0$, prove that:
$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9$$
5) For $a, b, c\geq 1$, prove that:
$$a^ab^bc^c\geq a^bb^cc^a$$
6) Maximise $x+\sqrt{1-x^2}$ where $|x|\leq 1$.
7) Maximise $\sin A+\sin B+\sin C$ where A, B, C are the angles in a triangle.
8) Let $a, b, c > 0$ and $abc=1$. Then show that: (IMO 1995)
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$