Graph Theory

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Definitions

- A **graph** is a set of **vertices** connected by **edges**. Two vertices connected by an edge are **adjacent**. A graph is **connected** if there is a path between any two vertices.
- The **degree** ($\deg(V)$) of a vertex V is the number of edges incident to V, with loops counted twice.
- **Simple:** Has no loops or multiple edges.
- **Planar:** *Can* be drawn (in a plane) with no edges crossing.
- **Bipartite:** Its vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other set.
- The **complete graph** on N vertices (K_N) is a graph in which every pair of distinct vertices is connected by one edge. The **complete bipartite graph** on sets of M and N vertices ($K_{M,N}$) is such that every pair of vertices in the two sets are adjacent.

Theorems

Denote V, E as the number of vertices and edges respectively. For a planar graph, let F be the number of faces (where the outside 'face' is also counted).

- $\sum \deg(V) = 2E$.
- **Euler Formula:** In a (finite) connected planar graph, $V - E + F = 2$.

Problems

1. **Handshaking Lemma:** In any graph, show that the number of odd vertices is even. (An *odd vertex* is a vertex of odd degree.)

2. In K_6 (the complete graph of six points) colour each edge red or blue. Is there always a *monochromatic triangle*; namely three vertices connected directly to each other by only red or only blue edges?

3. Show that in a simple planar graph, $E \leqslant 3V - 6$.

4. Show that K_5 is non-planar.

5. **Six Colour Theorem:** Given a simple connected planar graph, prove that each vertex can be coloured in one of six colours such that no two adjacent vertices are coloured the same colour.

6. There are three people, each of whose rooms need to be connected by wires to three appliances; a cooker, fridge and boiler. Is it possible to arrange the wires so that no wires cross?

7. A simple graph has 6 vertices, and has no triangles formed by three vertices. Can you draw such a graph with 9 edges? What about with 10 edges?

Further Problems

1. Use **Theorem 1** to count the number of edges in a K_n graph.

2. Show that every simple graph has two vertices of the same degree. Indeed, can you prove that two users of Facebook have the same number of friends?

3. A *tree* is a connected graph with no cycles. Show that any tree with at least two vertices is bipartite.

4. Prove that a graph is bipartite if and only if it has no odd cycles (a *cycle* is a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph).

5. Prove **Euler's Formula**. (Hint: try contracting edges)

6. (*IMO 1983*) The localities $P_1, P_2, \ldots, P_{1983}$ are served by ten international airlines $A_1, A_2, \ldots, A_{10}$. It is noticed that there is direct service (without stops) between any two of these localities and that all airline schedules offer round-trip flights. Prove that at least one of the airlines can offer a round trip with an odd number of landings.

Further Research

Graph Theory is a highly applicable topic, with a plethora of uses in Computer Science (solving Sudokus), Electronic Engineering (fitting circuits on one circuit board layer with planar graphs) and Transport (in the Underground).

If you wish to research into further topics included in this talk, here are a few elaborations from the problems discussed.

- Problem 2 is an insight into **Ramsey Theory**; finding 'order among disorder' in complete graphs. An exact formula for Ramsey numbers is unknown.

- Problem 4 and 6 naturally lead to **Kuratowski's Theorem.** In fact, K_5 and $K_{3,3}$ are special non-planar graphs. Are they *minimal*; that is, if you remove one edge, do they become planar? The wonderful Kuratowski's Theorem proves that *any* non-planar graph must contain a *subdivision* of K_5 or $K_{3,3}$ as a subgraph.

- Problem 5 should inspire you to learn about the (controversial) proof of the **Four Colour Theorem**.

- Problem 7 uses the **Cauchy-Schwarz Inequality**. It has a special meaning with vectors; the magnitude of the dot product of two vectors is less than or equal to the product of their magnitudes.

  Problem 7 also leads to one of my favourite theorems; **Turán's Theorem**. Firstly, can you generalise Problem 7 to n vertices? Now what if we replace triangle with K_p? In fact, Turán's Theorem states that the maximum number of edges that a simple graph on n vertices can have without containing K_p as a subgraph is:

$$\frac{n^2}{2}\left(1 - \frac{1}{p-1}\right)$$

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